17 Green's function for a second order ODE

The ultimate goal of this part of the course is to learn how it is possible, at least in principle, to solve the boundary value problem for the Poisson equation

$$-\Delta u = f, \quad \boldsymbol{x} \in D,$$

with the Dirichlet boundary conditions

$$u|_{\boldsymbol{x}\in\partial D} = h.$$

Here $\boldsymbol{x} = (x_1, \ldots, x_m) \in \mathbf{R}^m$, and the dimension of the space, of course, is important for how to proceed. In this section I will consider the case $m = 1$, and hence the domain D inevitably becomes an interval (a, b), which for simplicity I take to be $(0, 1)$, and the Poisson equation turns into a second order elementary ODE:

$$-cu'' = f, \quad x \in (0, 1), \tag{17.1}$$

(here c is a constant) with Type I boundary conditions, which I take to be homogeneous

$$u(0) = u(1) = 0. \tag{17.2}$$

Clearly, I do not need any special methods to solve this problem, since I can always integrate twice the equation and determine u up to two arbitrary constants, which, in their turn, can be determined from the boundary conditions. I, however, choose a somewhat more complicated way to attack this problem, which can be used in many other situations, opposite to direct integration, which I can do here.

It is useful to keep in mind a physical interpretation of problem (17.1), (17.2): It is the stationary distribution of the temperature inside an insulated rod with the internal energy sources described by f and with the temperature fixed at the ends at zero (one can certainly consider other possible boundary conditions, but see also below). A slightly more general situation occurs when the rod is not homogeneous, i.e., the properties to transfer the heat change from point to point. In this case the equation takes the form

$$-\big(c(x)u'\big)' = f, \quad x \in (0, 1),$$

which I will also consider in some detail.

17.1 Analogies with linear algebra

As in the case of Sturm–Liouville problem, it is beneficial first to consider a problem from linear algebra. Specifically, let $\boldsymbol{x} \in \mathbf{R}^n$ be an unknown vector that satisfies the equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

with square matrix $\boldsymbol{A}$, and given vector $\boldsymbol{b} \in \mathbf{R}^n$. We all know that this problem has a unique solution if and only if matrix $\boldsymbol{A}$ is invertible, and the solution is given by

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b},$$

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where $\boldsymbol{A}^{-1}$ is the inverse of matrix $\boldsymbol{A}$.

Now let me look into this solution in somewhat more details. First, I introduce the standard basis of $\mathbf{R}^n$: $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$, where, as usual, $\boldsymbol{e}_j$ is the vector with 1 at the j-th position and zeroes everywhere else. Any vector $\boldsymbol{b}$ can be written as a linear combination of these vectors:

$$\boldsymbol{b} = b_1\boldsymbol{e}_1 + \ldots + b_n\boldsymbol{e}_n.$$

Hence my solution can be written as

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} = \boldsymbol{A}^{-1}(b_1\boldsymbol{e}_1 + \ldots + b_n\boldsymbol{e}_n) = b_1\boldsymbol{A}^{-1}\boldsymbol{e}_1 + \ldots + b_n\boldsymbol{A}^{-1}\boldsymbol{e}_n.$$

That is, if I denote $\boldsymbol{x}_j$ (this is a vector, not a coordinate!) the solution to the problem

$$\boldsymbol{A}\boldsymbol{x}_j = \boldsymbol{e}_j,$$

then the solution to my original problem can be found as a linear combination of $\{\boldsymbol{x}_j\}_{j=1}^n$:

$$\boldsymbol{x} = b_1\boldsymbol{x}_1 + \ldots + b_n\boldsymbol{x}_n.$$

In words, to solve the problem $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, I may first find the n solutions to $\boldsymbol{A}\boldsymbol{x}_j = \boldsymbol{e}_j$, which are *the responses* of the operator $\boldsymbol{A}$ to the j-th unit basis vector, and then invoke *the principle of superposition*, which follows from the linearity of my problem, claiming that $\boldsymbol{x} = b_1\boldsymbol{x}_1 + \ldots + b_n\boldsymbol{x}_n$. If I need to consider a different right hand side, say, $\boldsymbol{q} = (q_1, \ldots, q_n)$ instead of $\boldsymbol{b}$, then the solution is $\boldsymbol{x} = q_1\boldsymbol{x}_1 + \ldots + q_n\boldsymbol{x}_n$. Problem solved.

To emphasize the analogies, let me rewrite problem (17.1), (17.2) as

$$Lu = f,$$

where L is the differential operator plus the boundary conditions. Formally, I am looking for something that would allow me to write

$$u = L^{-1}f.$$

Moreover, I also would like to understand what is the analogy here for the vectors $\boldsymbol{e}_j$ from the finite dimensional problem, and for the *special* solutions $\boldsymbol{x}_j$ that would allow me to construct solutions for *any* function f. The answers to these questions are given in the next section.

17.2 Green's function

The key difference with the linear algebra problem is that my f is an element of an infinite dimensional space, which is defined on the uncountable number of points in the interval $(0, 1)$, hence the index j must be replaced with a *continuous* variable ξ that takes all the values in $(0, 1)$. I also need something that would look like the principle of superposition. But now I already know (see the previous section!) that for any continuous f I can write

$$f(x) = \int_0^1 f(\xi)\delta(\xi - x)\,\mathrm{d}\xi = \int_0^1 f(\xi)\delta(x - \xi)\,\mathrm{d}\xi, \tag{17.3}$$

i.e., I represent f as a linear (continuous) combination of its own values at the points ξ with delta functions concentrated at ξ on the interval $(0, 1)$. In more detail, $\delta(x - \xi)$ plays the role of $\boldsymbol{e}_j$, $f(\xi)$

is "the coordinate" of f at point ξ, and the summation sign is replaced with the integral. Therefore, my hope, based on the analogy with the linear algebra case, that if I am able to solve problem (17.1), (17.2) with $\delta(x-\xi)$ instead of f, then the solution to the original problem can be found as a "linear combination" of f with this solution. Since the (family of) solutions to my problem with the right hand side given by the delta function play such an important role, I introduce

Definition 17.1. *The function $G(x;\xi)$, that depends on the real variable x and parameter ξ, that solves the problem*

$$-cG''(x;\xi) = \delta(x-\xi), \quad G(0;\xi) = G(1;\xi) = 0,$$

is called Green's (or response) function of the boundary value problem (17.1), (17.2).

The word "response" in the definition comes from the fact that Green's function physically is a reaction of the system for the unit heat energy source applied at the point $\xi \in (0,1)$. Following my intuition, I claim that if I know G then I can solve (17.1), (17.2) as

$$u(x) = \int_0^1 f(\xi)G(x;\xi)\,\mathrm{d}\xi, \tag{17.4}$$

using again the same principle of superposition (this is my L^{-1} from the previous subsection).

As a heuristic argument consider the following line of reasonings (not a proof!): Let (17.4) be true. Then, by plugging this expression into the equation and using (17.3) to rewrite f, I get

$$-c\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^1 f(\xi)G(x;\xi)\,\mathrm{d}\xi = \int_0^1 f(\xi)\delta(x-\xi)\,\mathrm{d}\xi,$$

which implies (if I am allowed to switch the order of integration and differentiation)

$$\int_0^1 f(\xi)\left(-cG''(x;\xi) - \delta(x-\xi)\right)\,\mathrm{d}\xi = 0,$$

which is true, due to the definition of G. The boundary conditions are also satisfied and hence my representation of the solution indeed works.

Example 17.2. So, let me actually find Green's function in the simplest possible case. I have, once again,

$$-cG'' = \delta(x-\xi), \quad G(0;\xi) = G(1;\xi) = 0.$$

I get

$$G'(x;\xi) = -\frac{1}{c}\int \delta(x-\xi)\,\mathrm{d}s = -\frac{1}{c}\chi(x-\xi) + A,$$

where χ is the Heaviside function. Integrating one more time, I get

$$G(x;\xi) = -\frac{1}{c}\rho(x-\xi) + Ax + B,$$

where ρ is the so-called ramp function, which is the integral of χ and defined as

$$\rho(x-\xi) = \begin{cases} 0, & x \leq \xi, \\ x-\xi, & x \geq \xi. \end{cases}$$

Now I can determine the constants A and B from the boundary conditions. The first boundary condition implies that $B = 0$. The second one yields (note that $1 \geq \xi$)

$$-(1-\xi)/c + A = 0 \implies A = \frac{1-\xi}{c}.$$

Therefore, I can write my final answer as

$$G(x;\xi) = \frac{(1-\xi)x - \rho(x-\xi)}{c} = \begin{cases} (1-\xi)x/c, & x \leq \xi, \\ (1-x)\xi/c, & x \geq \xi. \end{cases}$$

Now my solution for an arbitrary right hand side can be written as

$$u(x) = \int_0^1 f(\xi)G(x;\xi)\,\mathrm{d}\xi = \frac{1}{c}\int_0^x f(\xi)(1-x)\xi\,\mathrm{d}\xi + \frac{1}{c}\int_x^1 f(\xi)(1-\xi)x\,\mathrm{d}\xi. \tag{17.5}$$

Let me check that my formula actually works. Take, e.g., $f(x) = 1$, then the solution is

$$u(x) = \frac{1}{c}(1-x)\int_0^x \xi\,\mathrm{d}\xi + \frac{1}{c}x\int_x^1 (1-\xi)\,\mathrm{d}\xi = \frac{(1-x)x^2}{2c} + \frac{(1-x)^2x}{2c} = \frac{x-x^2}{2c}.$$

I will leave it as an exercise to find the same answer by direct integration of the problem (17.1), (17.2).

Exercise 1. Check directly that (17.5) solves problem (17.1), (17.2), by finding u'' using the Leibnitz formula to differentiate integrals that depend on x.

The last example shows several important properties of Green's function. In particular, this function is continuous, but not continuously differentiable (in the usual, not *generalized* sense), see Fig. 1. Its derivative has a jump of magnitude $-1/c$ at the point $x = \xi$. This function satisfies the boundary conditions and the equation at any point except $x = \xi$. These properties in general are enough to determine Green's function for more general problems (which are beyond this course). Note also that the found Green's function is symmetric: $G(x;\xi) = G(\xi;x)$, which physically means that the heat source applied at the point ξ feels the same wave at the point x as it would feel at the point ξ if applied at the point x (mathematically, this is actually the manifestation of the fact that my differential operator is self-adjoint).

Can we always find Green's function? Again, experience from linear algebra tells us that not all the problems of the form $\boldsymbol{Ax} = \boldsymbol{b}$ can be solved. The same is true for the differential operators. Consider the following example.

Example 17.3. Let me change the boundary conditions for Type II:

$$-cG''(x;\xi) = \delta(x-\xi), \quad x \in (0,1), \quad \xi \in (0,1), \quad G'(0;\xi) = G'(1;\xi) = 0.$$

Acting exactly as in the previous example I find again

$$G(x;\xi) = -\frac{1}{c}\rho(x-\xi) + Ax + B.$$

The first boundary condition implies that $A = 0$. However, the second boundary condition yields

$$-1/c + A = 0,$$

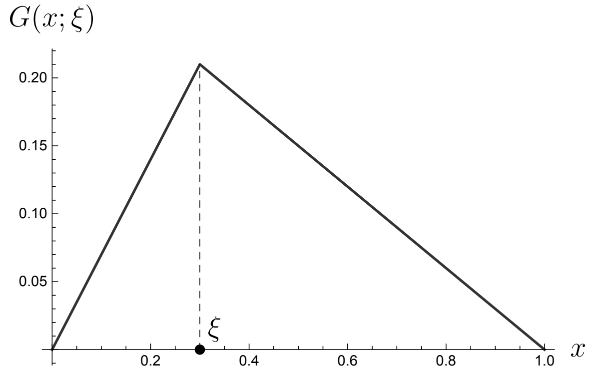

Figure 1: The graph of Green's function for problem (17.1), (17.2).

and hence I cannot come up with a suitable A.

The reason for this to happen is that the Neumann boundary value problem

$$-cu'' = f, \quad u'(0) = u'(1) = 0$$

does not have a *unique* solution. Indeed, take $f(x) = 1$ and conclude that there is no solution at all. On the other hand, take $f(x) = x - 1/2$ and conclude that there are infinitely many solutions. This gives at least some reason to understand why in this particular case there exists no Green's function. Actually, a deeper analysis of the problem shows that Green's function exists if and only if the corresponding boundary value problem with inhomogeneous right hand side (i.e., $f \neq 0$) has the unique solution and if and only if the only solution to the corresponding boundary value problem with $f = 0$ is the zero function (again compare with the conditions for the existence of the inverse matrix $\boldsymbol{A}^{-1}$).

Example 17.4. To finish this section, let me consider the problem with variable heat conductivity. Specifically, I am looking to solve

$$-\big(c(x)u'\big)' = f(x), \quad 0 < x < 1, \quad u(0) = u'(1) = 0,$$

where c is a continuous and positive on $[0, 1]$ function. Following the general ideology, I consider the problem

$$-\big(c(x)G'(x;\xi)\big)' = \delta(x-\xi), \quad 0 < x, \xi < 1, \quad G(0;\xi) = G'(1;\xi) = 0.$$

Integrating once implies that

$$c(x)G'(x;\xi) = -\chi(x-\xi) + A,$$

which can be used with the right boundary condition to infer that $A = 1$. Dividing by c and integrating again, I find that

$$G(x;\xi) = -\int_0^x \frac{\chi(s-\xi)\,\mathrm{d}s}{c(s)} + \int_0^x \frac{\mathrm{d}s}{c(s)} + B,$$

and the left boundary condition implies that $B = 0$. Finally I notice that my solution can be written as

$$G(x;\xi) = -\int_\xi^x \frac{\chi(s-\xi)\,\mathrm{d}s}{c(s)} + \int_0^x \frac{\mathrm{d}s}{c(s)} = \begin{cases} \displaystyle\int_0^x \frac{\mathrm{d}s}{c(s)}, & x \leq \xi, \\ \displaystyle\int_0^\xi \frac{\mathrm{d}s}{c(s)}, & \xi \leq x. \end{cases}$$

Note that the jump of the derivative of G at the point $x = \xi$ is $-1/c(\xi)$.

As usual, the solution to the original problem is therefore

$$u(x) = \int_0^1 f(\xi) G(x; \xi) \, \mathrm{d}\xi.$$

17.3 Solutions to the exercises